CLASA a XI-a

Problemă propusă ptr. O.L.M. 2010

1.02.2010

Dacă $A \in \mathcal{M}_4(\mathbb{R})$ cu proprietatea $A^4 = O_4$, atunci există $B \in \mathcal{M}_4(\mathbb{R})$ astfel încât $(I_4 + A)^3 = B^2$

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Soluție. Voi căuta pe B sub forma

(3p) $B = I_4 + xA + yA^2 + zA^3$

(1p) $$\begin{cases} B^2 = I_4 + x^2A^2 + y^2A^4 + z^2A^6 + 2xA + 2yA^2 + 2zA^3 + \\ \quad + 2xyA^3 + 2xzA^4 + 2yzA^5 = \\ = I_4 + 2xA + (x^2+2y)A^2 + (2z+2xy)A^3. \\ (A^4 = A^5 = A^6 = O_4) \end{cases}$$

Voi determina x, y, z astfel încât:

(2p) $$\begin{cases} 2x = 3,\ x^2 + 2y = 3,\ 2z + 2xy = 1 \text{ (comparând cu} \\ (I_4+A)^3) \end{cases}$$

Obținem $x = \frac{3}{2}$ $y = \frac{3}{8}$, $z = -\frac{1}{16}$ Deci

(1p) există $B = I_4 + \frac{3}{2}A + \frac{3}{8}A^2 - \frac{1}{16}A^3$ a.î.

$B^2 = (I_4 + A)^3$.

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Să se calculeze limitele

a) $\lim\limits_{x\to 0} \dfrac{(x+a^x)^x-1}{(x+b^x)^x-1}$, $a,b\in(0,\infty)-\{1\}$, $be\neq 1$ (3p)

b) $\lim\limits_{x\to 0} \dfrac{(x+a^x)^x-b^{x^2}}{(x+b^x)^x-a^{x^2}}$ (4p), $a,b\in(0,\infty)-\{1\}$, $be\neq a$

Gh. Andrei

Soluție

a)

$$\lim_{x\to 0} \frac{e^{x\ln(x+a^x)}-1}{e^{x\ln(x+b^x)}-1} = \lim_{\substack{x\to 0\\ u\to 0\\ v\to 0}} \frac{\frac{e^u-1}{u}\cdot x\ln(x+a^x)}{\frac{e^v-1}{v}\cdot x\ln(x+b^x)}$$

Calculăm: $\lim\limits_{x\to 0}\dfrac{\ln(x+a^x)}{\ln(x+b^x)} = \lim\limits_{x\to 0}\dfrac{\ln(1+x+a^x-1)}{\ln(1+x+b^x-1)} =$

$$= \lim_{\substack{x\to 0\\ f\to 0\\ g\to 0}} \frac{\frac{\ln(1+f)}{f}\cdot (x+a^x-1)}{\frac{\ln(1+g)}{g}\cdot(x+b^x-1)}$$ și $$\lim_{x\to 0}\frac{x+a^x-1}{x+b^x-1} = \lim_{x\to 0}\frac{1+\frac{a^x-1}{x}}{1+\frac{b^x-1}{x}} =$$

$$= \frac{1+\ln a}{1+\ln b}$$

b)

$$\lim_{x\to 0}\frac{\left(\frac{x+a^x}{b^x}\right)^x-1}{\left(\frac{x+b^x}{a^x}\right)^x-1} = \lim_{x\to 0}\frac{e^{x\ln\left(\frac{x+a^x}{b^x}\right)}-1}{e^{x\ln\left(\frac{x+b^x}{a^x}\right)}-1} = \lim_{\substack{x\to 0\\ u\to 0\\ v\to 0}}\frac{\frac{e^u-1}{u}\cdot u}{\frac{e^v-1}{v}\cdot v}$$

Calculăm $\lim\limits_{x\to 0}\dfrac{\ln\left(\frac{x+a^x}{b^x}\right)}{\ln\left(\frac{x+b^x}{a^x}\right)} = \lim\limits_{x\to 0}\dfrac{\ln\left(1+\frac{x+a^x-b^x}{b^x}\right)}{\ln\left(1+\frac{x+b^x-a^x}{a^x}\right)} =$

$$= \lim_{x\to 0}\frac{x+a^x-b^x}{x+b^x-a^x} = \lim\frac{1+\frac{a^x-1}{x}-\frac{b^x-1}{x}}{1+\frac{b^x-1}{x}-\frac{a^x-1}{x}} = \frac{1+\ln\frac{a}{b}}{1+\ln\frac{b}{a}} = \frac{\ln\frac{ae}{b}}{\ln\frac{be}{a}}$$

P) Cls. a XI-a OLM

Problema 1

(2p) a) Arătați că: $\left(1+\frac{\ln a}{n}\right)^n \geq 1+\ln a$, $\forall n \in \mathbb{N}^*$, $\forall a \geq 1$.

(1p) b) Dem. că: $a-1 \geq \ln a$, $\forall a \geq 1$.

(4p) c) Dem. că: $\lim_{n\to\infty}\left(\sqrt[n]{1}+\sqrt[n]{2}+\sqrt[n]{3}+\dots+\sqrt[n]{n}-n\right)=\infty$

(Nelu Chichirim)

Soluție: (2p) a) Bernoulli $\Rightarrow \left(1+\frac{\ln a}{n}\right)^n \geq 1+\cancel{n}\cdot\frac{\ln a}{\cancel{n}}=1+\ln a$.

b) Trecem la limită în ineg. de la pct. a $\Rightarrow$

(1p) $\Rightarrow \lim_{n\to\infty}\left(1+\frac{\ln a}{n}\right)^n \geq \lim_{n\to\infty}(1+\ln a) \Rightarrow$

$\Rightarrow e^{\ln a} \geq 1+\ln a \Rightarrow \boxed{a \geq 1+\ln a}$ ced

(4p) c) Cf pct. b $\Rightarrow \sqrt[n]{k}-1 \geq \ln\sqrt[n]{k}$ (1p), $(\forall)\, k \geq 1$.

$\Rightarrow \underbrace{\sum_{k=1}^{n}(\sqrt[n]{k}-1)}_{S_n} \geq \sum_{k=1}^{n}\frac{\ln k}{n}$ (1p) $\Rightarrow S_n \geq \frac{\ln 1+\ln 2+\dots+\ln n}{n}$

$\frac{\ln 1+\ln 2+\dots+\ln n}{n} \to \infty$ cu Crit. Cesaro-Stolz (2p)

$\Rightarrow S_n \xrightarrow{n} \infty$

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3) Fie $X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$X^6 = X \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot X \quad \underline{(1p)}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} c & a & b \\ f & d & e \\ i & g & h \end{pmatrix} = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \quad \underline{(2p)}$$

$$\det X = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ b & c & a \end{vmatrix} =$$

$$= (a+b+c)(a^2+b^2+c^2-ab-ac-bc) =$$

$$= \frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

cum $\det X^5 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \Rightarrow (\det X)^5 = 1$

$$\Rightarrow \det X = 1 \Rightarrow (a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] = 2 \quad \underline{(2p)}$$

$a, b, c \in \mathbb{Z} \Rightarrow \begin{cases} a+b+c=1 \\ (a-b)^2+(b-c)^2+(c-a)^2=2 \end{cases}$ sau $\begin{cases} a+b+c=2 \\ (a-b)^2+(b-c)^2+(c-a)^2=1 \end{cases}$ fals

$\Rightarrow (1,0,0), (0,1,0), (0,0,1)$ $\underline{(1p)}$

finalizare $\underline{(1p)}$